§ 2 Limits of Sequences

2.1 Definition

Definition 2.1.1 (Informal)

Let $\{a_n\}$ be a sequence of real numbers.

If n is getting larger and larger, a_n is getting closer and closer to $L \in \mathbb{R}$,

then we say L is the limit of the sequence a_n and we denote it by $\lim_{n\to\infty} a_n = L$.

Example 2.1.1

$\lim_{n\to\infty} \frac{1}{n} = 0$.

$\lim_{n\to\infty} (-1)^n$ does NOT exist.

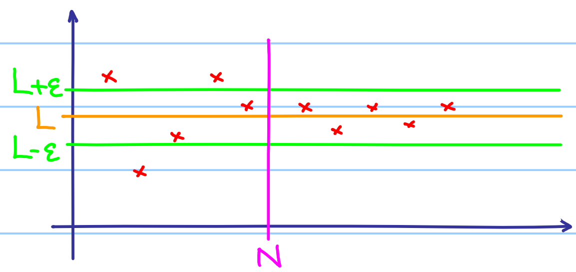
$\lim_{n\to\infty} 2^{n-1}$ does NOT exist.

Definition 2.1.2 (ε-definition)

Let $\{a_n\}$ be a sequence of real numbers and $L \in \mathbb{R}$.

L is said to be the limit of the sequence a_n if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ $\forall n \geq N$.



Meaning: No matter how small ε you give me,

I can always find a $N \in \mathbb{N}$ s.t. the tail (a_n with $n \geq N$) of sequence

lies in the ε-tunnel (ε-neighborhood of L)

Theorem 2.1.1

1) If $a_n = k$ $\forall n \in \mathbb{N}$ (constant sequence), then $\lim_{n\to\infty} a_n = k$.

2) If $k > 0$ and $a_n = n^{-k} = \frac{1}{n^k}$, then $\lim_{n\to\infty} a_n = 0$.

Remark: It seems that (1) and (2) are obvious, but we need to check the ε-definition, which is hard.

2.2 Algebraic Properties of Limits

Theorem 2.2.1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ (very important assumption),

then

1) $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L+M$

2) $\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L-M$

3) $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM$

4) If $M \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$

Example 2.2.1

Find $\lim_{n\to\infty} \frac{2}{n}+3$

Logically:

① $\lim_{n\to\infty} 2 = 2$, $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} \frac{2}{n} \overset{\text{By (3)}}{=} (\lim_{n\to\infty} 2)(\lim_{n\to\infty} \frac{1}{n}) = 2\cdot 0 = 0$

② $\lim_{n\to\infty} \frac{2}{n} = 0$, $\lim_{n\to\infty} 3 = 3$, so $\lim_{n\to\infty} \frac{2}{n}+3 \overset{\text{By (1)}}{=} \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3 = 0+3 = 3$

But what we write:

$$\lim_{n\to\infty} \frac{2}{n}+3 = \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3$$
$$= 0+3$$
$$= 3$$

Example 2.2.2

Find $\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$

$\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$ (We cannot use 4, why?)

$= \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{2-\frac{4}{n}}$ (Now, we can use 4!)

$= \frac{\lim_{n\to\infty} 1+\frac{3}{n^2}}{\lim_{n\to\infty} 2-\frac{4}{n}}$

$= \frac{1}{2}$

Exercise 2.2.1

Find $\lim_{n\to\infty} \frac{3n+1}{n^2-2n}$, $\lim_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ (if exist)

Answer : $\lim_{n\to\infty} \frac{3n+1}{n^2-2n} = 0$, $\lim_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ does NOT exist.

Any observation ?

Basically, we are comparing the degrees of the numerator and the denominator.

Conclusion :

If $p(x)$ and $q(x)$ are polynomials,

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \neq 0$ ($\deg p(x) = m$)

$q(x) = b_k x^k + a_{k-1} x^{k-1} + \cdots + b_1 x + b_0$ with $b_k \neq 0$ ($\deg q(x) = k$)

then

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \infty & \text{if } m > k \\ \frac{a_m}{b_k} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Following this idea :

Example 2.2.3

Find $\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$

$\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$ ← roughly deg = 1

$= \lim_{n\to\infty} \frac{3-\frac{1}{n}}{\sqrt{4+\frac{2}{n}}}$

$= \frac{3}{2}$

Example 2.2.4

Find $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$

$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$

$= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$= 0$

Example 2.2.5

Find $\lim_{n\to\infty} \frac{2^n}{n}$.

Question: Can we say $\frac{2^n}{n} = \frac{1}{n} \cdot 2^n$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} \frac{2^n}{n} = 0$?

Absolutely NOT !

Since $\lim_{n\to\infty} 2^n$ does NOT exist, property (3) cannot be applied !

2.3 Monotonic Sequence Theorem

Defintion 2.3.1

Let $\{a_n\}$ be a sequence of real numbers.

(i) $\{a_n\}$ is said to be bounded above if $\exists M > 0$ s.t. $a_n \leq M$ — called an upper bound

(ii) $\{a_n\}$ is said to be bounded below if $\exists M > 0$ s.t. $a_n \geq M$ — called a lower bound

(iii) $\{a_n\}$ is said to be bounded if $\exists M > 0$ s.t. $|a_n| \leq M$ (i.e. $-M \leq a_n \leq M$)

bounded = both bounded above and below

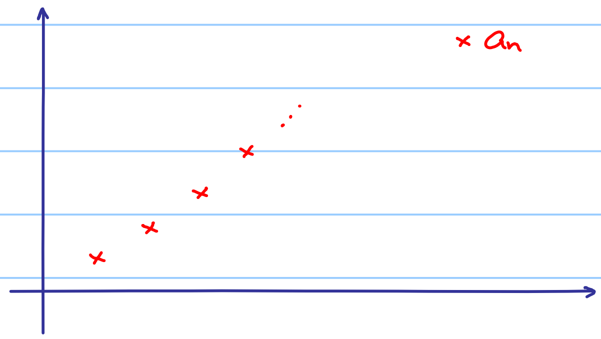
(iv) $\{a_n\}$ is said to be monotonic increasing if $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$

(v) $\{a_n\}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

Geometrical meaning:

$\{a_n\}$ is bounded above by M

But it may happen that a sharper bound M'



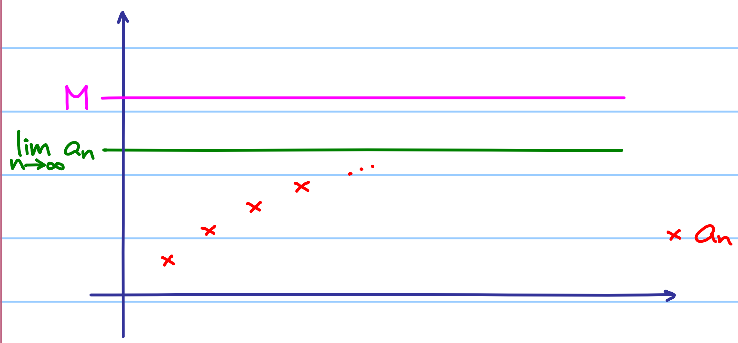
$\{a_n\}$ is monotonic increasing.

Combining together:

Theorem 2.3.1 (Monotone Convergence Theorem)

If $\{a_n\}$ is bounded above (resp. below) and monotonic increasing (decreasing), then $\lim\limits_{n\to\infty} a_n$ exists.

Geometrical meaning:



Caution:

$\{a_n\}$ is bounded above by M,

but $\lim\limits_{n\to\infty} a_n$ is NOT necessary to be M.

Example 2.3.1

Let $\{a_n\}$ be a sequence of real numbers defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{a_n}{1+a_n} \quad (n \geq 1)$$

Does $\lim_{n\to\infty} a_n$ exist ?

i) Claim: $\{a_n\}$ is monotonic increasing

(Note: From the construction of the sequence, $a_n \geq 0 \ \forall n \in \mathbb{N}$)

Prove the statement "$a_{n+1} \geq a_n$" by induction:

Step 1: $a_2 - a_1 = \left(1 + \frac{a_1}{1+a_1}\right) - a_1 = \frac{3}{2} - 1 = \frac{1}{2} > 0$

Step 2: Assume $a_{k+1} \geq a_k$ for some $k \in \mathbb{N}$

$$\begin{aligned} a_{k+2} - a_{k+1} &= \left(1 + \frac{a_{k+1}}{1+a_{k+1}}\right) - \left(1 + \frac{a_k}{1+a_k}\right) \\ &= \frac{a_{k+1}}{1+a_{k+1}} - \frac{a_k}{1+a_k} \\ &= \frac{a_{k+1} - a_k}{(1+a_{k+1})(1+a_k)} \geq 0 \end{aligned}$$

ii) $\{a_n\}$ is bounded above by 2.

$\therefore$ By Monotone Convergence Theorem, $\lim_{n\to\infty} a_n$ exists (But, what is the value ?)

Let $\lim_{n\to\infty} a_n = A$

Note that $a_{n+1} = 1 + \frac{a_n}{1+a_n}$, taking limit on both sides

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 + \frac{a_n}{1+a_n} = 1 + \frac{\lim_{n\to\infty} a_n}{1 + \lim_{n\to\infty} a_n}$$

$$A = 1 + \frac{A}{1+A}$$

$$A^2 + A - 1 = 0$$

$$A = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2} \text{ (rejected)}$$

Note: the limit is NOT 2.

Constant e :

Consider a number $(1+\frac{1}{m})^n$ that depends on m and n and then

1) fix m, say $m=100$, n is getting larger and larger.

$n=10$ $\quad$ $n=100$ $\quad$ $n=1000$ $\quad \to +\infty$

$(1+\frac{1}{m})^n = 1.01^{10}$ $\quad$ $(1+\frac{1}{m})^n = 1.01^{100}$ $\quad$ $(1+\frac{1}{m})^n = 1.01^{1000}$ $\quad \to +\infty$

2) fix n, say $n=100$, m is getting larger and larger.

$m=10$ $\quad$ $m=100$ $\quad$ $m=1000$ $\quad \to +\infty$

$(1+\frac{1}{m})^n = 1.1^{100}$ $\quad$ $(1+\frac{1}{m})^n = 1.01^{100}$ $\quad$ $(1+\frac{1}{m})^n = 1.001^{100}$ $\quad \to 1$

How about setting $m=n$ and let them become larger and larger ?

$(1+\frac{1}{n})^n \to ?$ as $n \to +\infty$ (i.e. limit exists ?

something between $+\infty$ and 1 ??)

$n=10$ $\quad$ $n=100$ $\quad$ $n=1000$ $\quad \to +\infty$

$(1+\frac{1}{n})^n = 1.1^{10}$ $\quad$ $(1+\frac{1}{n})^n = 1.01^{100}$ $\quad$ $(1+\frac{1}{n})^n = 1.001^{1000}$

≈ 2.59374 $\quad$ ≈ 2.70481 $\quad$ ≈ 2.71692 $\quad \to 2.71828\cdots$

limit exists and call it e .

How to prove ?

Idea :

Let $a_n = (1+\frac{1}{n})^n \quad \forall n \in \mathbb{N}$.

1) Prove $\{a_n\}$ is monotonic increasing ;

2) Prove $\{a_n\}$ is bounded above by 3 .

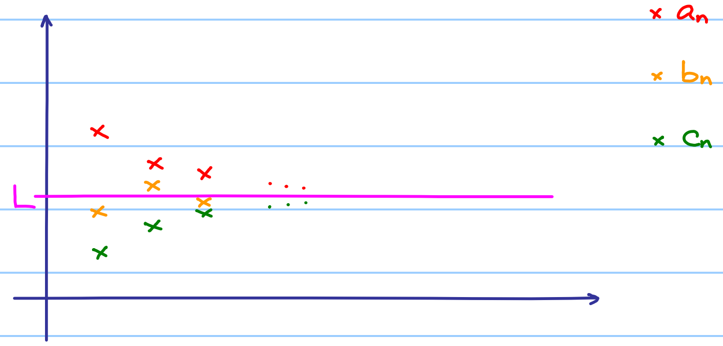
2.4 Sandwich Theorem

Theorem 2.4.1 (Sandwich Theorem)

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers.

If $a_n \le b_n \le c_n$ $\forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Geometrical meaning:



Example 2.4.1

Find $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

Note: $0 \le \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \le \dfrac{1}{2\sqrt{n}}$ $\forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \dfrac{1}{2\sqrt{n}} = 0$

By sandwich theorem, $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

Example 2.4.2

Find $\lim\limits_{n\to\infty} \frac{1}{n}\sin n$

Note: $-\frac{1}{n} \le \frac{1}{n}\sin n \le \frac{1}{n}$ $\forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} -\frac{1}{n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

By sandwich theorem, $\lim\limits_{n\to\infty} \frac{1}{n}\sin n = 0$.

Exercise 2.4.1

Prove that $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n^2} = 0$

Hint: $-1 \le (-1)^n \le 1$.

Theorem 2.4.2

Let $\{a_n\}$ be a sequence of real numbers.

$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$.

proof:

"$\Leftarrow$" Suppose that $\lim_{n\to\infty} |a_n| = 0$.

Note that $-|a_n| \le a_n \le |a_n| \quad \forall n \in \mathbb{N}$, and $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$,

by the sandwich theorem, $\lim_{n\to\infty} a_n = 0$.

"$\Rightarrow$" Suppose that $\lim_{n\to\infty} a_n = 0$.

Then $\lim_{n\to\infty} a_n^2 = (\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} a_n) = 0\cdot 0 = 0$

Note that $|a_n| = \sqrt{a_n^2}$

$$\therefore \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \sqrt{a_n^2} \overset{(*)}{=} \sqrt{\lim_{n\to\infty} a_n^2} = \sqrt{0} = 0$$

(*) is true because of $\sqrt{x}$ is a function that is continuous at 0.

By using the above result, we obtain a result concerning a product of two sequences:

Theorem 2.4.3

If $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} a_n b_n = 0$.

proof:

Note
- $-|a_n| \le a_n \le |a_n| \quad \forall n \in \mathbb{N}$
- $\{b_n\}$ is bounded $\Rightarrow \exists M > 0$ s.t. $|b_n| \le M$ (i.e. $-M \le b_n \le M$) $\forall n \in \mathbb{N}$.

$\therefore -M|a_n| \le a_n b_n \le M|a_n| \quad \forall n \in \mathbb{N}$

$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$

$\Rightarrow \lim_{n\to\infty} -M|a_n| = \lim_{n\to\infty} M|a_n| = 0$

$\therefore$ By the sandwich theorem, $\lim_{n\to\infty} a_n b_n = 0$.